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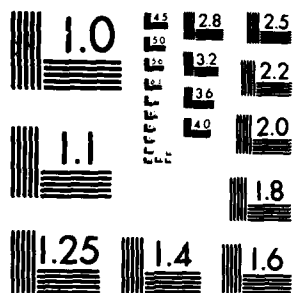
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CONNECTION FOR WAVE MODULATION

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UNIVERSITY OF WISCONSIN-MADISON  
MATHEMATICS RESEARCH CENTER

CONNECTION FOR WAVE MODULATION

R. E. Meyer and J. F. Painter\*

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ABSTRACT

A new approach is described to the connection of wave amplitudes across the turning points and singular points of second-order, linear, analytic, ordinary differential equations which can describe the modulation of physical waves or oscillators. The general class of singular points thereby defined (Section 2) contains many irregular ones of greater complexity than have been accessible before; however, genuine coalescence of singular points is not here considered. The asymptotic connection formulae are shown to result directly from the branch structure of the singular point (Section 3); indeed, to a first approximation, they reflect merely the gross, local branch structure. The proof (Section 4) relates the local structure of the solutions at the singular point to the asymptotic wave structure by a limit process justified by symmetry bounds.

AMS (MOS) Subject Classifications: 34E20, 41A60, 30E15

Key Words: Schroedinger equation, oscillator modulation, WKB-connection, turning point.

Work Unit Number 2 - Physical Mathematics

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## SIGNIFICANCE AND EXPLANATION

This work concerns the modulation of waves or oscillating systems, which pervade all the science and engineering disciplines. Modulation occurs when waves travel through an inhomogeneous material in which the local propagation velocity differs from place to place, but the differences are small over a distance of only a wavelength -- a very common case in the sciences and engineering. The resulting change to the waves is mostly gradual, but occasionally drastic, as at a shadow-boundary, where oscillation turns into decay and quiescence over just a few wavelengths. When this phenomenon can be analyzed via an ordinary differential equation, such a boundary is called a transition point.

At first, only the simplest transition points representing the most typical shadow boundaries were studied. But then some phenomena, such a wave reflection and scattering cross-sections, came to be traced to hidden transition points that become visible only when real distance (or time) is embedded in its complex plane. When the material properties vary in a general manner, (which can often be observed only incompletely) the hidden transition points can have arbitrarily complex structure. The following represents a new, more direct and more general approach to the connection of waves and shadows across transition boundaries. It aims to furnish a basis for more efficient wave scattering calculations.

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## CONNECTION FOR WAVE MODULATION

R. E. Meyer and J. F. Painter<sup>\*</sup>

### 1. Introduction

The semi-classical Schroedinger equation

$$(1) \quad \epsilon^2 \frac{d^2 w}{dz^2} + p w(z) = 0$$

with small parameter  $\epsilon$  and analytic coefficient function  $p(z)$  is central to a vast class of oscillation and wave modulation problems in physics and other sciences. Particular interest, especially for scattering theory, attaches to the "WKB" problem of connecting the wave-approximations to solutions across roots or singular points of  $p(z)$ . The following introduces a connection method which is simpler and more general than any advanced before [Zwaan 1919, Langer, 1931, Painter and Meyer 1981]. Simplification and clarification of connection theory is, in fact, the whole objective of the study to be reported, and generalization was used only as a help towards it.

One reason why this objective has proved elusive over the generations may be that the general, second-order, linear differential equation, of which (1) is the normal form, encompasses too many disparate phenomena. The present study focuses on only those forms of (1) which are genuine Schroedinger equations in the sense that they can describe the modulation of physical waves or oscillators. This subclass is characterized in Section 2 in terms of its admissible (turning-point and) singular-point structure. To attempt only one step at a time, moreover, genuine coalescence of singular points is excluded.

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This leaves a large class of singular points, none the less, because the potential functions  $p(z)$  of (1) in the sciences must be defined, it not by speculation, then by measurement, in which case they can be known only imperfectly. In addition, it has long been recognized that scattering matrices may depend greatly on singular points of  $p(z)$  away from the real axis of time or space, in which case there are scant physical grounds for restricting their nastiness. The characterization of  $p(z)$  cannot therefore be very specific and must include arbitrarily irregular points of (1). Certainly, multi-valued functions  $p(z)$  must be the norm, rather than the exception. All the same, modulation implies a certain structure (Section 2).

The multi-valuedness of  $p(z)$  in (1) implies that the solutions  $w(z)$  must normally be multi-valued, and the main thesis to be propounded is that this multi-valuedness is the source of the connection problem and that the asymptotic connection formulae solving it are a direct manifestation of the branch structure of the potential  $p(z)$  at the singular point. This fundamental view of connection is adopted also by Olver [1974, pp. 481, 482] for regular and isolated singular points. Our objective is to show how it can be extended to large classes of very irregular ones and what new insights into the nature of connection emerge therefrom.

To obtain ammunition for this thesis, local solution representations near the singular point have been developed in a companion paper [Meyer and Painter 1982, hereafter referred to as IPM] and are summarized in Section 3. They focus on a particular fundamental system  $y_s, y_m$  of (1) in which  $y_m$  has a milder singularity than  $y_s$  which, in turn, contains no additive multiple of  $y_m$ . This makes the pair a characteristic representation of the branch structure of the singular point. The representation is a local one, in the

first place, but turns out to have a striking two-scale structure: in the framework of a natural, independent variable  $x$  (Section 2), the solutions do not depend on the complex parameter  $\epsilon$  in (1) separately, but only on the variables  $x$  and  $\epsilon x$ . A key property of the representations is that they are global in the oscillation variable  $x$ , even though only local in the modulation variable  $\epsilon x$ , and can be extended to similar bounds on a loss of symmetry of the solutions with increasing distance from the singular point.

To document the thesis, these symmetry bounds are shown in Section 4 to admit a class of limit processes in which both  $|x| \rightarrow \infty$  and  $|\epsilon x| \rightarrow 0$  and furthermore, the symmetry of  $y_s$  and  $y_m$  is essentially preserved. In other words, asymptotic approximation of WKB-type characterized by dominance and recessivity is shown to become available before local structure has been lost. Existence of such limit processes translates immediately local information on the structure of  $y_s$  and  $y_m$  at the singular point into information on the multivaluedness of their asymptotic wave-representation. But, the latter information is what connection theory seeks.

To make our point, it will be sufficient to document it in the generic case, leaving aside the "Frobenius exceptions" [Olver 1974 pp. 150] which involve logarithmic branch points for regular points and consequent loss of the symmetry bounds, in the general case. The validity of the final connection formulae (Section 4) for the exceptional cases is strongly indicated by the results of Painter and Meyer [1981], but the approach adopted in the following seems unsuited for a simple proof. It is also sufficient to document our point at the instance of the first asymptotic approximation, which connects the wave-amplitudes, and attention will be restricted to it. A more exhaustive description may be thought desirable for the sake of completeness, and most of all, error bounds are desirable. It may be noted



that the representations used are obtained by the standard method of Volterra integral equations, which is precisely the method leading to effective error bounds [Olver 1974]. The sketch of the representation method given in Section 3 makes clear that such bounds would emerge in terms of pointwise and integral bounds on a certain irregularity function  $\phi(\epsilon x)$  arising in the characterization of the singular point (Section 2). In the general case, however, that function is barely specified and can tend to zero with  $\epsilon x$  arbitrarily slowly, so that it appears doubtful whether the bounds would give much satisfaction to the numerical analyst. Most of all, however, we suspect that the present proof of connection may not remain the simplest one for long, in which case present attention to error bounds and higher approximations may be premature.

## 2. Modulation Equations

Equation (1)

$$\epsilon^2 \frac{d^2 w}{dz^2} + p(z, \epsilon) w(z) = 0$$

is one of a large family of normal forms of the general, linear, second-order differential equation and constructive general statements are difficult in such an indefinite frame. By contrast,

$$dx/dz = ip^{1/2}/\epsilon$$

defines the Lionville-Green or WKB or Langer variables  $x$  and  $\epsilon x$  based on the local wavelength or period, which have long been recognized as the natural ones for wave modulation. Physical specifications, e.g. for scattering, relate directly to them and if  $z$  differs substantially from  $x$ , it can at best measure distance in legal units. The natural formulation of physical problems of wave modulation is therefore in terms of  $x$  or  $\epsilon x$ , from the start, which will avoid the extraneous difficulty of describing the global

transformation between  $\epsilon x$  and  $z$ , which has no physical significance and can be a very complicated, multi-valued map.

The exclusion of coalescence, in order to confine attention to one singular point at a time, restricts not their total number, but only how fast they can approach each other as  $\epsilon \rightarrow 0$ . When this is not fast enough to introduce genuine coalescence, a rescaling [Meyer and Guay 1974] permits the elimination of the main  $\epsilon$ -dependence from  $\mu(z;\epsilon)$  and therefore, not much generality is lost by ignoring the residual  $\epsilon$ -dependence. for simplicity,  $p(z)$  will accordingly be taken independent of  $\epsilon$  in what follows.

The main property distinguishing the wave modulation equations among the larger class of equations (1) is that the natural variable  $x$  must be definable, for otherwise, not even the concepts of wavelength or period could exist for (1). An additional requirement arises as follows. If  $p^{1/2}$  be non-integrable at a singular point, then that point is seen to correspond to no  $x \in \mathbb{C}$  and hence, represents not a genuine singularity but a device for re-interpreting radiation conditions as a singular point in the  $z$ -planes. Such a device has been used at times in quantum mechanics, but is excluded here to concentrate on the class of genuine singularities of modulation. For that class, the singular point of (1) must correspond to a definite point  $x$ . Without loss of generality, both will be identified with this origin.

For an effective formulation of this notion of the most general wave modulation equation (short of coalescence), it should be expressed in terms of the natural variable  $x$ . Accordingly, the following is based on the premise that a branch  $r(x)$  of  $p^{1/4}$  is definable as an analytic function on a punctured neighborhood of  $x = 0$  which is a Riemann surface including the interval  $(-\pi, 2\pi)$  of  $\arg x$  so that

$$(2) \quad i \, dz/dx = \epsilon \, r^{-2}$$

is integrable at  $x = 0$ . (In conventional, turning-point terminology, such a Riemann surface element comprises three adjacent Stokes sectors.)

When the Schroedinger equation (1) is transformed to the natural variable by, say, simply setting

$$w(z) = y(x) ,$$

it takes the form

$$(3) \quad y'' + 2r^{-1}r' y' - y = 0 ,$$

which shows that the wave development is controlled by the "modulation function"

$$r'(x)/r(x) = (i\epsilon/2)d(p^{-1/2})/dz ,$$

rather than by the potential function  $p(z)$  directly. This illuminates why it has long been known that the singular points of (1) should really include the roots of  $p(z)$  ("turning points"). It is also seen that the modulation function has a particular structure: since  $p(z) = r^4$  is a function of  $z$  independent of  $\epsilon$ , it follows from (2) that  $\epsilon x$  is also such a function and in turn, that  $xr'/r$  depends on  $x$  and  $\epsilon$  only through the product  $\epsilon x$ . A secondary hypothesis to be now adopted, because it simplifies the theory of connection, is that a limit of  $xr'/r$  as  $\epsilon x \rightarrow 0$  can be identified,

$$(4) \quad xr'/r \rightarrow \gamma \in \mathbb{C} \text{ as } \epsilon x \rightarrow 0$$

uniformly in the Riemann surface sector  $\Delta$  of  $\epsilon x$  in which  $xr'/r$  is defined locally near  $\epsilon x = 0$ . These two hypotheses also define the framework of the analysis of [IPM]. A statement equivalent to (4) is that the (fourth root of the) potential  $r(x)$  can be written in the form

$$(5) \quad p^{1/4} = r(x) = x^\gamma \rho(\epsilon x)$$

where  $\rho(\xi)$  is a function analytic on the Riemann surface element  $\Delta$  with the property

$$(6) \quad (\xi/\rho) d\rho/d\xi = : \phi(\xi) \rightarrow 0 \text{ as } \xi \rightarrow 0$$

uniformly in  $\Delta$ , because  $\phi(\epsilon x) = x r'/r - \gamma$ .

To make the structure of the theory more readily apparent, it will help to abbreviate the notation by the convention that a function symbol such as  $g(x)$  is always understood to denote a function of both  $x$  and  $\epsilon x$ . By contrast a Greek symbol such as  $\psi(\xi)$  will always denote a function of  $\xi = \epsilon x$  only. If such a function has the property (6),

$$(\xi/\psi) d\psi/d\xi \rightarrow 0 \text{ as } \xi \rightarrow 0$$

uniformly in  $\Delta$ , then it will be called mild; it implies that  $\psi(\xi)$  varies near  $\xi = 0$  less than any non-zero real power of  $\xi$ :

$$\forall \nu > 0, \quad |\xi^\nu \psi^{\pm 1}| \rightarrow 0 \text{ as } \xi \rightarrow 0.$$

In particular, the limit  $\gamma$  postulated in (4) is thus seen from (5) to represent the exponent of the "nearest power" of  $x$  in the (fourth root of the) potential, and the basic integrability premise defining physical Schroedinger equations implies

$$\operatorname{Re} \gamma < \frac{1}{2}.$$

The general class of singular points of Schroedinger equations thus defined includes very irregular ones, in addition to all the turning points covered in the literature [Painter and Meyer 1981]. For Langer's [1931] class of fractional turning points,

$$z^{2\gamma/(2\gamma-1)} [p(z)]^{1/2}$$

is analytic and nonzero at  $z = 0$ ,

$$\phi(\xi) = \sum_{n=1}^{\infty} \gamma_n \xi^{(1-2\gamma)n}$$

and the solutions of (1) and (3) are approximable in terms of Bessel functions [Langer 1931, Olver 1977]. For other singular points, however, no simple approximands in terms of Bessel functions appear likely. Local

approximations have been constructed in [IPM] to provide support for the present study even under the vague assumptions just sketched, which admit functions  $\phi(\xi)$  of arbitrary multivaluedness and approaching zero as  $\xi \rightarrow 0$  more slowly than any definite function. Similarly, the coefficient functions  $p(z)$  in (1) here admitted can be of great complexity, especially when several irregular singular points are present, and a useful global description appears unlikely in the general case. Locally, however, the class of potential functions  $p(z)$  can be described by

$$z^{-1} \int_0^z [p(t)/p(z)]^{1/2} dt \rightarrow 1 - 2\gamma \text{ as } z \rightarrow 0.$$

As indicated in the Introduction, the conceptual key to connection lies in the two-variable structure of the Schroedinger equation emerging from (2) to (4). It should not have surprised us as much as it did, for it is already apparent in (1) that the independent variable plays two distinct physical roles. The first term in (1) represents the oscillatory mechanism and its independent variable is clearly  $z/\varepsilon$ , with local wavelength as natural unit, prompting the transformation to

$$(7) \quad x = \frac{1}{\varepsilon} \int_0^z \sqrt{p(t)} dt.$$

By contrast,  $p(z)$  represents the potential and its variation, which is not dependent on the presence of waves, and the role of  $z$  in it is therefore a different one. The formulation sketched in this Section adds the insight that the dependence on the modulation variable  $z \leftrightarrow \varepsilon x = \xi$  enters into the Schroedinger equation (3) only through a relatively minor term

$$\phi(\varepsilon x) = x r' / r - \gamma$$

in the modulation function  $r'/r$ . This function  $\phi$  has been called "irregularity function" in [IPM] because  $\phi = 0$  characterizes the regular singular points.

The critical role of this two-variable structure will emerge in Section 4 which is devoted to a proof that connection across singular points is a mathematical process local in  $\epsilon x$ , even though asymptotic in  $x$ . This is, perhaps, the main new insight gained by extending the fundamental view of connection of Olver [1974] to irregular singular points. It explains why a merely local definition of  $\phi(\epsilon x)$  - - and thereby, of the potential and of the Schroedinger equation - - on a Riemann surface element

$$\Delta = \{\epsilon x: -\pi < \arg(\epsilon x) < 2\pi, 0 < |\epsilon x| < E \text{ for some } E > 0\}$$

turns out sufficient. (For notational convenience,  $E$  is adjusted so that  $\phi(\epsilon x)$  is analytic up to the rim of the element and hence, bounded on  $\Delta$ .) In a shortwave limit  $\epsilon \rightarrow 0$ , that entails little restriction on the corresponding domain  $D$  of  $x$ . On account of the two-variable structure, moreover, a shortwave limit must be a limit  $\epsilon x \rightarrow 0$ . Hindsight, of course, makes all of this appear foreshadowed in the structure of (1), where the first, oscillation term is defined globally in  $z/\epsilon$ , even if the potential  $p(z)$  be defined only locally.

### 3. Branch Structure

It will help to summarize now the result of [IPM] used in the proof of connection in Section 4 and to indicate their motivation. If a limit  $\epsilon \rightarrow 0$  is taken at fixed  $x$ , then  $\phi(\epsilon x) \rightarrow 0$ , by (5), and (3) approaches a form of Bessel's equation (which observation started Langer's [1931] work). Its singular point is regular with Frobenius exponents  $0$  and  $1-2\gamma > 0$ . If  $1-2\gamma$  is not an integer, that implies solutions  $f_s(x)$  and  $x^{1-2\gamma} f_m(x)$  with entire functions  $f_s, f_m$  (and  $f_s(0) = f_m(0) = 1$ ), which turn out to depend only on  $x^2$ . Integer values of  $\frac{1}{2} - \gamma$  correspond to the Frobenius exceptions for which only  $f_m(x)$  is entire.

It is plausible that (3) may have an analogous fundamental system  $(y_s, y_m)$  when  $\epsilon \neq 0$ , which displays the branch structure of the irregular singular point most clearly. If  $y_m/y_s \rightarrow 0$  as  $x \rightarrow 0$ , then  $y_s$  has there the stronger singularity and it appears natural to call it the stronger solution and  $y_m$ , the milder. Such a system has been constructed [IPM] to obtain a representation of the branch structure at the general irregular point of wave modulation and in particular, to find out what replaces the entire functions  $f_s, f_m$  and to explore how departure from entirety can be characterized. The underlying idea emerges most simply in the following construction of  $y_s(x)$  for  $\frac{1}{2} > \operatorname{Re} \gamma > -\frac{1}{2}$ .

Since (3) can be written  $(r^2 y')' = r^2 y$ , a simple Volterra equation associated with it is

$$y'(x) = [r(x)]^{-2} \int_0^x [r(v)]^2 y(v) dv \quad (8)$$

$$y(x) = 1 + \int_0^x y'(v) dv .$$

By (5), a simple iterative approach to (8) is by a sequence  $\{b_n(x)\}$  such that

$$\frac{db_{n+1}}{dx} = \int_0^x \left[ \frac{\rho(\epsilon v)}{\rho(\epsilon x)} \right]^2 \left( \frac{v}{x} \right)^{2\gamma} b_n(v) dv , \quad (9)$$

$$b_0 \equiv 1, \quad b_n(0) = 0 \text{ for } n > 1 ,$$

and since  $\rho(\xi)$  is a mild function, it can be estimated at the expense of a small power. It emerges readily in this way, by estimation of  $b'_n$  from (9) and (6) and thence,  $b_n$ , recursively, that

$$\begin{aligned} b_n(x) &= \beta_n(\epsilon x) (x/2)^{2n} , \\ |\beta_n(\xi)| &\leq k'_n = \Gamma(-s)/[n! \Gamma(n-s)] , \\ s &= -\operatorname{Re} \gamma - \frac{1}{2} + \sup_{u \in (0,1)} |\phi(u\xi)| . \end{aligned} \quad (10)$$

Therefore, if  $|\varepsilon x| \leq E(\gamma)$  chosen to assure  $s < 0$ , then the rapid decrease of  $k'_n$  with  $n$  documents a majorant series assuring the convergence of

$\sum b_n$  to a solution  $y_s(x)$  of (3) analytic on the Riemann sector  $D$ .

Since  $y_s(0) = 1$ ,  $\sum b_n(x)$  generalizes Frobenius' entire function  $f_s(x)$ , but the 'coefficients'  $\beta_n$  are generally multi-valued functions of  $\varepsilon x$ .

Observe that (10) suggests  $y_s = \sum b_n$  tends to an even function of  $x$  in some sense as  $\varepsilon x \rightarrow 0$ . This can be made more precise by applying the same approach to the estimation of  $|b'_n(x) + b'_n(x \exp(-\pi i))|$  and thence,

$|b_n(x) - b_n(x \exp(-\pi i))|$  to show [IPM] that

$$|b_n(x) - b_n(xe^{-\pi i})| \leq \delta_s(|\varepsilon x|)^n k'_n |x/2|^{2n}$$

where

$$(11) \quad \delta_s(|\xi|) \rightarrow 0 \text{ as } |\xi| \rightarrow 0.$$

These bounds still decrease fast enough with  $n$  to be summed to a bound on the degree of oddness of  $y_s(x)$  in terms of the modified Bessel function

$I_\nu(z)$  [Olver 1974, p. 60]: For  $\frac{1}{2} > \operatorname{Re} \gamma > -\frac{1}{2}$  and  $x, x \exp(-\pi i)$  in  $D$  and  $|\varepsilon x| \leq E(\gamma)$ ,

$$(12) \quad |y_s(x) - y_s(xe^{-\pi i})| \leq \delta_s(|\varepsilon x|) \Gamma(-s) |x/2|^{2+s} I_{-s}(|x|).$$

Thus  $y_s(x)$  approaches evenness as  $|\varepsilon x| \rightarrow 0$  uniformly on compact subsets of the cut  $x$ -plane. For fixed  $\varepsilon x$ , on the other hand,  $y_s(x)$  may lose its evenness exponentially fast as  $|x|$  increases.

A good representation of the milder solution  $y_m(x)$  of (3) depends on identification of the exact function generalizing the factor  $x^{1-2\gamma}$  of  $f_m(x)$ . It turns out to be just the function  $z(x)$  defined by (2), indeed [IPM],

$$(13) \quad z(x) = x^{1-2\gamma} \zeta(\varepsilon x)$$

with a mild function  $\zeta(\xi)$  such that

$$(14) \quad (1-2\gamma)\rho^2 \zeta \rightarrow -i\varepsilon \text{ as } \xi \rightarrow 0.$$



Then  $y_m(x)/z(x) = \hat{y}(x)$  satisfies a differential equation related to (3) and can be constructed for all  $\text{Re } \gamma < \frac{1}{2}$  by an iteration paralleling that just sketched to obtain [IPM] a representation

$$(15) \quad y_m(x) = z(x) \left[ 1 + \sum_{n=1}^{\infty} \alpha_n(\epsilon x) (x/2)^{2n} \right]$$

with bounds

$$|\alpha_n(\xi)| \leq k_n = \Gamma(m)/[n! \Gamma(m+n)] ,$$

where  $m = 3/2 - \text{Re } \gamma - \delta_2(|\epsilon x|) > 0$  for  $|\epsilon x| < \text{another } E(\gamma)$ , since

$\delta_2(|\xi|) \rightarrow 0$  as  $|\xi| \rightarrow 0$ . Of course,  $\zeta(\xi)$  and  $\alpha_n(\xi)$  are generally multivalued functions, but  $\alpha_n(0)$  is defined and nonzero, so that

$y_m/z = \hat{y}(x)$  also tends to an even function as  $\epsilon x \rightarrow 0$ . An oddness bound is obtained by an estimate paralleling that indicated above:

Theorem 4 [IPM]. For  $x$  and  $x e^{-\pi i}$  in  $D$  and  $|\epsilon x| < E(\gamma)$ ,

$$|\hat{y}(x) - \hat{y}(x e^{-\pi i})| \leq \delta_m(|\epsilon x|) \Gamma(m) |x/2|^{2-m} I_m(|x|)$$

$$\text{and } \delta_m(|\xi|) \rightarrow 0 \quad \text{as } |\xi| \rightarrow 0.$$

The same comment therefore applies to  $y_m/z = \hat{y}$  as follows (12).

No similarly simple approach has been found yet for the stronger solution  $y_s$  for  $\text{Re } \gamma < -\frac{1}{2}$ , where the simple Volterra equation (8) can, by (5), be used only at the price of a regularization of the first integral in (8). A stronger solution in the sense indicated is defined only up to an additive multiple of the milder solution, which is undesirable for a fundamental system displaying clearly the branch structure of the singular point. The regularization adopted in [IPM] avoids that additive multiple and constructs a stronger solution of the form

$$(16) \quad y_s(x) = 1 + \sum_{n=1}^{\infty} \beta_n(\epsilon x) (x/2)^{2n}$$

for all  $\operatorname{Re} \gamma < \frac{1}{2}$ , but the estimates of  $|\beta_n|$  are more laborious, and less close, than for  $\frac{1}{2} > \operatorname{Re} \gamma > -\frac{1}{2}$ . Indeed,  $\beta_n(0)$  is found to exist for all  $n$  only when  $\frac{1}{2} - \operatorname{Re} \gamma$  is not an integer and the near-evenness of  $y_s(x)$  holds only for them:

Theorem 5 [IPM]. For non-integer  $\frac{1}{2} - \operatorname{Re} \gamma > 0$ ,  $x$  and  $xe^{-\pi i}$  in  $D$  and sufficiently restricted  $|\varepsilon x|$ ,

$$|y_s(x) - y_s(xe^{-\pi i})| \leq C(\gamma) \delta_s(|\varepsilon x|) |x/2|^{2+s} I_{-s}(|x|),$$

with  $s = -\operatorname{Re} \gamma - \frac{1}{2} + \sup_{u \in (0,1)} |\phi(u\xi)| > 0$  and  $\delta_s(|\xi|) \rightarrow 0$  as  $|\xi| \rightarrow 0$ .

#### 4. Connection

If  $y(x)$  satisfies (3), then  $W(x) = r(x)y(x)$  satisfies

$$(17) \quad W'' = (1 + r''/r)W$$

and by (4)

$$(18) \quad r''/r = x^{-2}[\gamma(\gamma-1) + \phi(2\gamma-1 + \phi + \xi\phi'/\phi)]$$

so that  $|r''/r|$  is integrable along paths in  $D$  bounded away from  $x = 0$ .

This confirms [Olver 1974, p. 222] existence of a fundamental "WKB" solution pair

$$(19) \quad W_+(x) = a(x)e^x, \quad W_-(x) = b(x)e^{-x}$$

with functions  $a(x)$ ,  $b(x)$  analytic on  $D$  and bounded for large  $|x|$

(provided, of course,  $\varepsilon x = \xi \in \Delta$  so that  $\phi$  and  $\xi\phi'$  are bounded). This is the fundamental system of (1) most strikingly describing the asymptotic wave character (undamped on the lines where  $x$  is pure imaginary) of the solutions.

The "amplitude functions"  $a$ ,  $b$  are determined only up to a constant factor, but apart from that, the decay of  $|r''/r|$  at large  $|x|$  suffices [Olver 1974, p. 223, 224] to assure limits for  $a(x)$  and  $b(x)$  as  $|x| \rightarrow \infty$

with  $\arg x$  any integer multiple of  $\pi$ . Those limits are the well-known wave amplitudes in the first-order WKB-approximations to the solutions of (1).

Since any solution of (17) must be a linear combination of  $W_+$  and  $W_-$ ,

$$(20) \quad r(x)y_m(x) = \tilde{a}_m(x)e^x + \tilde{b}_m(x)e^{-x}$$

and the same holds with subscript  $s$  in place of  $m$ , and supposing they can be normalized satisfactorily, then  $\tilde{a}_m, \dots, \tilde{b}_s$  are similarly analytic and bounded for large  $|x|$ . Since the lefthand side of (20) has been seen in Section 3 to be multivalued, not all of  $\tilde{a}_s, \dots, \tilde{b}_m$  can be entire, and symmetry makes it plausible that all of them will usually turn out to be multivalued. This prompts the question

$$\tilde{a}_m(\infty) - \tilde{a}_m(\infty e^{2\pi i}) = ?$$

which is, in fact, a connection question for WKB coefficients [Olver 1974, p. 481].

In view of the many contexts in which connection is important, it is natural that many different forms of the connection problem are found in the literature, but most of them can be related to each other with little work, and a treatise on connection for simple turning points is found in Chapter 13 of [Olver 1974]. In any case, the problem turns on relating the respective limits which represent the WKB coefficients on different domains, and when it is recognized that those domains correspond, in the frame of the natural variable, to sheets of the Riemann surface of the solution, the form of the connection question just arrived at is seen to be a natural one.

By contrast to the functions  $a(x), b(x)$  first mentioned,  $\tilde{a}_m(x)$  and  $\tilde{b}_m(x)$  are normalized implicitly by the normalization of  $y_m(x)/z(x) = \hat{y}(x)$  and this turns out to introduce an  $\epsilon$ -dependence into the normalization of  $\tilde{a}_m, \tilde{b}_m$ . For fixed  $\epsilon \neq 0$ , moreover,  $|x|$  is bounded by  $E/\epsilon$  on the Riemann sector  $D$  on which the differential equation (1) has been defined,

and the connection question can therefore be posed only in the limit  $\epsilon \rightarrow 0$ . This aspect is discussed in the Appendix, where it is shown that the functions

$$\tilde{a}_m/(\rho\zeta) = a_m(x; \epsilon x) \quad \text{and} \quad \tilde{b}_m/(\rho\zeta) = b_m(x; \epsilon x) ,$$

rather than  $\tilde{a}_m$  and  $\tilde{b}_m$  themselves, are certain to have limits as  $\epsilon \rightarrow 0$  and  $|x| \rightarrow \infty$ . For an assuredly meaningful connection question, we should therefore rewrite the identity (20) as

$$(21) \quad r(x)[\rho(\xi)\zeta(\xi)]^{-1} y_m(x) = a_m(x) e^x + b_m(x) e^{-x}$$

(with explicit mention of the dependence of  $a_m, b_m$  on  $\xi = \epsilon x$  omitted to focus attention now on  $\arg x$ ) and ask  $a_m(\infty e^{2\pi i}) - a_m(\infty) = ?$

Since (21) is an identity in  $x$  on  $D$ , it holds equally at  $x \exp(-\pi i)$ , if that point is also in  $D$ . If  $\exp(-\pi i)$  be abbreviated by  $j$ , then since  $y_m = z\hat{y}$  and  $rz = x^{1-\gamma}\rho\zeta$ , by (5) and (13), the identity

$$(22) \quad \begin{aligned} [y(x) - y(jx)]x^{1-\gamma}e^{-|x|} &= [a_m(x) - j^{\gamma-1}b_m(jx)]e^{x-|x|} \\ &= [b_m(x) - j^{\gamma-1}a_m(jx)]e^{-x-|x|} \end{aligned}$$

also holds on  $D$ . Now let  $|\epsilon x| \rightarrow 0$ , but  $|x| \rightarrow \infty$  in such a way that the lefthand side of (22) still tends to zero. That this does indeed define a non-empty set of "intermediate limits", in the terminology of singular-perturbation theory [Eckhaus 1979], is a corollary of Theorem 4 because [Olver 1974, p. 435]

$$I_m(|x|) \sim |2\pi x|^{-1/2} e^{|x|} \quad \text{as} \quad |x| \rightarrow \infty$$

and, e.g.,  $|x| = |\log \delta_m(|\epsilon x|)|$  will serve.

For the choices  $\arg x = \pi$  and  $\arg x = 2\pi$ , respectively, such a limit of (22) yields

$$(23) \quad b_m(\infty e^{\pi i}) = j^{\gamma-1} a_m(\infty), \quad a_m(\infty e^{2\pi i}) = j^{\gamma-1} b_m(\infty e^{\pi i}) ,$$

whence

$$(24) \quad a_m(\infty e^{2\pi i}) - a_m(\infty) = 2i \sin(\gamma\pi) \quad b_m(\infty e^{\pi i}) ,$$

The choice  $\arg x = 0$  adds

$$(25) \quad a_m(\infty) = j^{\gamma-1} b_m(\infty e^{-\pi i})$$

to (23), whence the answer to the connection question for  $b_m$  is

$$(26) \quad b_m(\infty e^{\pi i}) - b_m(\infty e^{-\pi i}) = 2i \sin(\gamma\pi) a_m(\infty).$$

For non-integer  $\frac{1}{2} - \operatorname{Re} \gamma$ , a parallel argument for the stronger solution starts from the identity

$$r(x)y_s(x) = \tilde{a}_s(x)e^x + \tilde{b}_s(x)e^{-x}$$

analogous to (20) to deduce that the normalization  $y_s(0) = 1$ ,  $y'_s(0) = 0$  assures boundedness of

$$\tilde{a}_s(x)/\rho(\xi) = a_s(x) \quad \text{and} \quad \tilde{b}_s(x)/\rho(\xi) = b_s(x)$$

as  $\xi \rightarrow 0$  and leads via the identity (22) with  $\hat{y}$ ,  $m$  and  $1-\gamma$  replaced, respectively, by  $y_s$ ,  $s$  and  $\gamma$ , by the help of Theorem 5 to the same connection formulae (24), (26) for  $a_s$ ,  $b_s$  in the place of  $a_m$ ,  $b_m$ , because  $\sin[(1-\gamma)\pi] = \sin(\gamma\pi)$ . [It is this independence of subscript which makes (24), (26) more convenient for present purposes than various other relations, such as  $a_m(\infty \exp 2\pi i) = a_m(\infty) \exp(-2\gamma\pi i)$ , also implied by (23) and (25), or their counterparts for  $a_s$ ,  $b_s$  obtained by replacement of  $1-\gamma$  by  $\gamma$ .]

With appropriate interpretation, moreover, the same connection formulae relate  $\tilde{a}_m$ ,  $\tilde{b}_m$  and  $\tilde{a}_s$ ,  $\tilde{b}_s$ , respectively, because  $(\rho^2 \zeta)^{-1}$  tends to a definite limit as  $\xi \rightarrow 0$ , by (14), and by (6),

$$\frac{\rho(j\xi)}{\rho(\xi)} = \exp \int_{\xi}^{j\xi} \phi(\tau) \frac{d\tau}{\tau} = \exp \int_0^{-\pi} \phi(|\xi|e^{i\sigma}) i d\sigma + 1.$$

Since any solution  $y(x)$  of (3) is a linear combination of the fundamental pair  $(y_s, y_m)$ , the functions  $a(x)$  and  $b(x)$  in the representation

$$r(x)y(x) = a(x)e^x + b(x)e^{-x}$$

in terms of the fundamental pair  $(W_+, W_-)$  of (17) are linear combinations of  $\tilde{a}_a$ ,  $\tilde{a}_m$  and  $\tilde{b}_s$ ,  $\tilde{b}_m$ , respectively, and therefore satisfy (24) and (26) as well. In the limit  $\varepsilon \rightarrow 0$  and with interpretation appropriate to the

normalization of  $y(x)$ , the connection formulae (24) and (26) are therefore a general corollary of (3) under the two hypotheses of Section 2, at least, as long as  $\operatorname{Re} \gamma - \frac{1}{2}$  is not a negative integer.

# Appendix

The somewhat delicate issue of normalization for connection may be brought under control in two steps. For fixed  $\epsilon$ , the domain  $D$  of  $x$  on which  $r(x)$ , and hence also the differential equation (17), is defined is a Riemann sector of radius  $E(Y)/\epsilon$ . Let the particular functions  $a, b$  in (19) normalized in the manner of Olver [1974, pp. 220-222] be denoted by  $a_0, b_0$ . Then  $a_0[b_0] = 1$  and  $a'_0[b'_0] = 0$  at a point of  $\min_D[\max_D] \operatorname{Re} x$ , which depends on  $\epsilon$ , and thus  $a_0 = a_0(x; \epsilon)$ ,  $b_0 = b_0(x; \epsilon)$ , and the first step will be to confirm that their dependence on  $\epsilon$  weakens as  $\epsilon \rightarrow 0$ .

Since  $\phi(\xi)$  is analytic and bounded on the Riemann sector  $\Delta$  for  $0 < |\xi| \leq E(Y)$ , the same follows for

$$\begin{aligned} x^2 \psi(x) &= Y(Y-1) + (2Y-1)\phi(\xi) + \phi^2 + \xi \phi'(\xi) \\ &= x^2 r''/r \end{aligned}$$

in (18) because it tends to  $Y(Y-1)$  as  $\xi = \epsilon x \rightarrow 0$ , by (6). For the basic connection question of Section 4, it is sufficient to restrict the Riemann sector  $D$  to a disc of the same radius cut along the positive real axis of  $x$  for  $a_0(x; \epsilon)$ , and along the negative real axis, for  $b_0(x; \epsilon)$ . Olver's [1974, p. 221] variation function for  $a_0$  is

$$\Psi(x) = \int_{-|E/\epsilon|}^x |\psi(v)| dv$$

evaluated along progressive paths, and if such a path keeps distance  $R$  from the origin, then since  $x^2 \psi$  is bounded,

$$(A1) \quad \Psi = O(R^{-1}) \quad \text{as } R \rightarrow \infty$$

The same holds for the variation function for  $b_0$ , which differs only in that the lower limit is  $+|E/\epsilon|$ . The functions furnish [Olver 1974, p. 221] bounds

$$(A2) \quad |a_0(x; \epsilon) - 1|, \quad |a'_0(x; \epsilon)| \leq e^{\Psi(x)} - 1 = O(R^{-1})$$

and similarly, for  $b_0$ . If now  $|\epsilon_k| < |\epsilon_1|$ , then  $D(\epsilon_k) \supset D(\epsilon_1)$  and on

$D(\epsilon_i)$  the solution  $W_+$  normalized for  $D(\epsilon_i)$  is a linear combination of  $W_+$ ,  $W_-$  normalized for  $D(\epsilon_k)$ , thus

$$a_0(x; \epsilon_i) e^x = c_{ik} a_0(x; \epsilon_k) e^x + d_{ik} b_0(x; \epsilon_k) e^{-x}$$

with constants  $c_{ik}$ ,  $d_{ik}$  computed from the respective normalization and bounds to yield

$$a_0(x; \epsilon_i) = a_0(x; \epsilon_k) \{1 + O(\epsilon_i)\} + b_0(x; \epsilon_k) e^{-2(x+E/\epsilon_i)} O(\epsilon_i)$$

and on  $D(\epsilon_i)$ ,  $\text{Re}(x + E/\epsilon_i) > 0$ . As long as  $|x|$  is well bounded away from 0, therefore,  $a_0(x; \epsilon_i) = a_0(x; \epsilon_k) + O(\epsilon_i)$  as  $|\epsilon_i| \rightarrow 0$ , and similarly for  $b_0(x; \epsilon)$ , and by the bounds (A1), (A2),  $a_0$  and  $b_0$  tend to limits as  $|x| \rightarrow \infty$  in  $D(0)$ .

For the second step, note that the amplitude functions  $\tilde{a}_m$  and  $\tilde{b}_m$  in (20) are renormalizations of  $a_0$  and  $b_0$  so that

$$\tilde{a}_m(x; \epsilon) = A a_0(x; \epsilon), \quad \tilde{b}_m(x; \epsilon) = B b_0(x; \epsilon)$$

with coefficients  $A$ ,  $B$  possibly dependent on  $\epsilon$ . From (20), (5) and (13), therefore,

$$(A3) \quad x^{1-\gamma} y_m(x)/z(x) = x^{1-\gamma} \hat{y}(x) = (\rho\zeta)^{-1} [A a_0(x; \epsilon) e^x + B b_0(x; \epsilon) e^{-x}]$$

The differential equation for  $\hat{y}(x)$  is, by (2) and (3),

$$\hat{y}'' + 2(r'/r + z'/z) \hat{y}' = \hat{y}$$

and since the normalization to  $\hat{y}(0) = 1$ ,  $\hat{y}'(0) = 0$  recognized in (15) is independent of  $\epsilon$ ,  $\hat{y}$  inherits from  $(r'/r + z'/z)x$  the property that it depends on  $\epsilon$  only through  $\xi = \epsilon x$ . Like  $(r'/r + z'/z)x$ , moreover,  $\hat{y}$  depends continuously on  $\xi$  in  $\Delta$ , for fixed  $x \neq 0$ , and as  $\xi \rightarrow 0$ , by (4) and (13),  $r'/r + z'/z \rightarrow (1-\gamma)/x$  and the differential equation for  $\hat{y}$  becomes a form of Bessel's, with solution

$$(A4) \quad \lim_{\xi \rightarrow 0} \hat{y}(x) = \Gamma\left(\frac{3}{2} - \gamma\right) (x/2)^{\gamma-1/2} I_{1/2-\gamma}(x)$$

This shows  $x^{1-\gamma} \hat{y}(x)$  to tend to a well-defined function of  $x$  on  $D(0)$  as  $\xi \rightarrow 0$ , and since  $a_0(x; \epsilon)$  and  $b_0(x; \epsilon)$  have been shown to tend to limit



functions on  $D(0)$  as  $\varepsilon \rightarrow 0$ , it follows from (A3) that the functions  $A/(\rho\zeta)$  and  $B/(\rho\zeta)$  must tend to limits as  $\varepsilon \rightarrow 0$  and  $\xi \rightarrow 0$ ; of course, these limits might depend on the direction of approach, which is determined by  $\arg x$ .

In sum,

$$\frac{\tilde{a}_m}{\rho\zeta} = \frac{A}{\rho\zeta} a_0(x; \varepsilon), \quad \frac{\tilde{b}_m}{\rho\zeta} = \frac{B}{\rho\zeta} b_0(x; \varepsilon),$$

where  $A/(\rho\zeta)$ ,  $B/(\rho\zeta)$  have limits as  $\varepsilon \rightarrow 0$  and  $\xi \rightarrow 0$ , while the limits as  $\varepsilon \rightarrow 0$  of  $a_0$ ,  $b_0$  are defined on  $D(0)$  and tend there to limits as  $|x| \rightarrow \infty$ .

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ABSTRACT (continued)

they reflect merely the gross, local branch structure. The proof (Section 4) relates the local structure of the solutions at the singular point to the asymptotic wave structure by a limit process justified by symmetry bounds.

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